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On the regularity of the flow of analytic vector fields

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ABSTRACT

The flow of vector field X is the solution to the system of linear partial differential equations $\partial_t \mathbf{u}(t,z) = X(\mathbf{u}(t,z))$ with initial condition $\mathbf{u}(0,z) = z$. It is known that for an analytic vector field, its flow is given by a convergent Lie series. Recently, Carillo gave a quick and elementary proof of this fact by introducing a special family of norms. This paper gives an even more concise proof by directly estimating the formal series expansion and using a family of majorant functions studied by Lax.

INTRODUCTION

Let Ω be an open neighborhood of the origin $0 \in \mathbb{C}^n$, and let X be an analytic vector field on Ω, i.e., it is a partial differential operator of the form

$$X = \sum_{i=1}^{n} X_i(z)\partial_{z_i},$$

where each $X_i(z)$ is an analytic function of $z = (z_1, \dots, z_n)$ on Ω. The flow of X, denoted by $\Phi_X(t,z)$, is the solution of the initial value problem

$$\begin{cases} \partial_t \mathbf{u}(t,z) = X(\mathbf{u}(t,z)) \\ \mathbf{u}(0,z) = z. \end{cases} \tag{1.1}$$

The flow $\Phi_X(t,z)$ can be interpreted geometrically as follows: for any point $y \in \Omega$, the curve $\Phi_X(t,y)$ passes through the point y at $t = 0$ and whose tangent vector at $\Phi_X(t,y)$ is given by $X(\Phi_X(t,z))$.

It is a well-known result in the general theory of differential equations that the Cauchy problem (1.1) possesses a unique solution, and that this solution has the same regularity as the vector field X. In particular, if X is analytic, then so is $\Phi_X(t,z)$ in the variables (t,z). We formally state this as a theorem:

Theorem 1. *If X is an analytic vector field on Ω, then the Cauchy Problem* (1.1) *has a unique analytic solution $\Phi_X(t,z)$ in a neighborhood of the origin $(0,0) \in \mathbb{C}^{n+1}$. Moreover, this solution is given by the series*

$$\Phi_X(t,z) = \sum_{k=0}^{\infty} (X^k(z_1), \dots, X^k(z_n)) \frac{t^k}{k!}. \tag{1.2}$$

Here, X^k denotes iterations of X, i.e., for an analytic function $f(z)$, we define $X^0(f) = f$ and $X^{k+1}(f) = X(X^k(f))$ for $k \geq 0$.

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The classical proof of this theorem follows the arguments in proving the Cauchy-Kowalevsky Theorem (see, e.g., Gröbner 1960), as the former is simply a special case of the latter. It proceeds by first showing that (1.1) has a unique formal power series solution of the form $\mathbf{u}(t,z)=\sum_{k\geq 0}\mathbf{u}_k(z)t^k$, which is quite straightforward. This is followed by employing the Cauchy majorant technique to prove that this formal series converges near the origin (and thus defines an analytic function there). This second part of the proof constructs an auxiliary nonlinear partial differential equation (PDE) that is simpler than (1.1) and is easily shown to have an analytic solution (Cartan 1995). Another popular method of proof is via the Fixed Point Theorem, which also works even when the vector field has lesser regularity (see, e.g., Ilyashenko and Yakovenko 2008).

Recently, Carrillo (2021) offered another proof of this theorem by introducing a linear auxiliary PDE whose formal solution is precisely the series (1.2). Convergence was then proved using the concept of Nagumo norms, and finally, (1.2) was verified to satisfy the Cauchy problem (1.1). Carrillo claimed that his proof is quicker and more elementary because the auxiliary equation is linear (as opposed to the nonlinear auxiliary equation used in Cauchy's method of majorants). Furthermore, although his proof of convergence made use of the little-known Nagumo norm, the concept employed elementary notions in complex analysis that are accessible to undergraduate mathematics students.

In this short article, we present another proof that is even simpler, eliminating the need for an auxiliary equation. We instead make use of Lax's majorant series to directly estimate the expansion (1.2) and immediately establish its convergence. Our method relies only on elementary concepts of complex analysis, and unlike the fixed-point approach, it does not need to introduce a new norm in some Banach space of functions.

Proof of the Theorem

Unique existence of a formal solution. Since we seek an analytic solution $\mathbf{u}(t,z)$ to (1.1), let us assume a formal solution of the form $\mathbf{u}(t,z)=\sum_{k\geq 0}\mathbf{u}_k(z)t^k$, where the coefficients $\mathbf{u}_k(z)$ are unknown analytic functions defined near the origin. (For simplicity in notation, we will continue using $\mathbf{u}$ for the unknown function instead of the more common notation for the flow of X.)

The given initial condition gives $\mathbf{u}_0(z)=0$. To determine the other coefficients, we substitute the formal solution to (1.1), perform the formal differentiations, and compare coefficients of equal powers of t. We easily see that:

$$\mathbf{u}_1 = X(\mathbf{u}_0) = X(z)$$
$$2\cdot\mathbf{u}_2 = X(\mathbf{u}_1) = X^2(z),$$

and in general,

$$k\cdot\mathbf{u}_k = X(\mathbf{u}_{k-1}) = \frac{X^k(z)}{(k-1)!}.$$

All these imply that $\mathbf{u}_k = X^k(z)/k!$ for $k\geq 0$, that is, each $\mathbf{u}_k$ is determined recursively and uniquely. Note that this coincides with the expansion given in (1.2), which is actually a Lie series. Note further that since X is an analytic vector field on Ω, each $\mathbf{u}_k$ is also analytic there.

Lax's majorant function. The second part of the proof is devoted to proving that the obtained formal solution $\mathbf{u}(t,z)$ is convergent in some neighborhood of the origin. As in the classical proof or in the new one offered by Carrillo, convergence is the more challenging part of the proof. We now introduce the main tool in our proof.

Given two (possibly formal) power series $a(w)=\sum a_\alpha w^\alpha$ and $b(w)=\sum b_\alpha w^\alpha$, we say that *a is majorized by b* if $|a_\alpha|\leq b_\alpha$ for all multi-indices α. We denote this by $a(w)\ll b(w)$. The concept naturally extends to vector-valued functions if we define majorization component-wise.

The main tool in our proof of convergence is the series introduced by Lax (1953), namely,

$$\varphi(x)=\frac{1}{4S}\sum_{k=0}^{\infty}\frac{x^k}{(k+1)^2}. \qquad (2.1)$$

Here, the constant S is equal to $1+\frac{1}{4}+\frac{1}{9}+\cdots$, and was introduced by Lope and Tahara (2002) so that certain majorant relations will hold (as will be seen later). The series is easily seen to be convergent for $|x|\leq 1$ and thus defines an analytic function in this domain. The following lemma states some useful properties of φ (see also Lope and Tahara 2002).

Lemma 2. *The following majorant relations hold:*

(a) $\varphi^2\ll\varphi$

(b) $\varphi^{(m)}\varphi\ll\varphi^{(m)}$ *for any nonnegative integer* m

(c) *Given* $\varepsilon_0\in(0,1)$, *there exists a constant* $C>0$ *dependent only on* ε_0 *such that for any* $\varepsilon\in(0,\varepsilon_0]$,

$$\frac{1}{1-\varepsilon x}\cdot\varphi\ll C\varphi.$$

Proof. The proofs of (a) and (c) may be found in Lope and Tahara (2002), but we reproduce them here for completeness. To prove (a), we have to show that for each $k\geq 1$,

$$\frac{1}{4S}\sum_{l=0}^{k}\frac{1}{(l+1)^2(k-l+1)^2}\leq\frac{1}{(k+1)^2}.$$

The estimation is a bit mechanical: we first consider the square of the quantity $(l+1)^{-1}(k-l+1)^{-1}$ and later note that $\sum_{l=0}^{k}(l+1)^{-1}(k-l+1)^{-1}\leq\sum_{l=0}^{k}(l+1)^{-2}$. Note that by our choice of S, we have $S\geq\sum_{l=0}^{k}(l+1)^{-2}$.

We prove (b) by induction. From (a), the relation holds when $m=0$. Suppose now that it is true when $m=j$. Observe that

$$(\varphi^{(j)}\varphi)'=\varphi^{(j+1)}\varphi+\varphi^{(j)}\varphi'.$$

Since the coefficients of φ and its derivatives are all positive, it follows that

$$\varphi^{(j+1)}\varphi\ll(\varphi^{(j)}\varphi)'\ll\varphi^{(j+1)},$$

where we used the induction hypothesis in the last simplification. This completes the induction.

Finally, since $\varphi^2\ll\varphi$ from (a), we can prove (c) by showing that

$$\frac{1}{1-\varepsilon x}\ll K\varphi \qquad (2.2)$$

for some $K>0$ that is dependent only on ε_0. However, this can be deduced from the fact that for $0<\varepsilon_0<1$, the quantity $\varepsilon_0^k(k+1)^2$ tends to 0 as $k\to\infty$.

Remark 2.1. Note that (2.2) implies that analytic functions can be majorized by constant multiples of φ. For suppose that f is analytic in a neighborhood of $|z|\leq R_0$, for some $R_0>0$, and is bounded there by $A>0$. Then by Cauchy's inequality, we have

$$f(z) \ll \frac{A}{1-\frac{z}{R_0}} \ll \frac{A}{1-\frac{R}{R_0}\frac{z}{R}} 4S\varphi\left(\frac{z}{R}\right),$$

where the last relation holds because $4S\varphi \gg 1$. Applying (2.2) with $\varepsilon = R/R_0$ and using Lemma 2(a), we finally see that

$$f(z) \ll K'\varphi\left(\frac{z}{R}\right),$$

for some $K' > 0$ that depends on f and on the ratio R/R_0. The latter can be set to be at most $\frac{1}{2}$ (or any fraction less than 1) by restricting the range of R. This will allow us to use a uniform constant K' for all values of $R \leq \frac{1}{2}R_0$.

Convergence of the formal solution. Since X is an analytic vector field on Ω, there exist positive constants A, R such that $X_i(z) \ll A\varphi(z/R)$ for all $i = 1, \ldots, n$. Here, for brevity, we use the notation $|z| = z_1 + \cdots + z_n$, which should not cause any confusion because majorization of power series is not concerned about the modulus of z.

Now since $z_i \ll 4R \cdot 4S\varphi(|z|/R)$ by definition of φ, we can majorize $X(z)$ as follows:

$$\begin{aligned} X(z) &\ll \sum_{i=1}^{n} X_i(z) \frac{\partial}{\partial z_i}\left[16RS\varphi\left(\frac{|z|}{R}\right)\mathbf{1}\right] \\ &\ll \sum_{i=1}^{n} 16AS\varphi\left(\frac{|z|}{R}\right)\varphi'\left(\frac{|z|}{R}\right)\mathbf{1} \\ &\ll 16nAS\varphi'\left(\frac{|z|}{R}\right)\mathbf{1}, \qquad (2.4) \end{aligned}$$

because of Lemma 2(b). Here, we denoted by $\mathbf{1}$ the $n \times 1$ vector whose entries are all 1. We then apply X repeatedly to the majorant in (2.4) to obtain

$$X^k(z) \ll 16RS\left(\frac{nA}{R}\right)^k \varphi^{(k)}\left(\frac{|z|}{R}\right)\mathbf{1},$$

which finally allows us to majorize the formal solution as follows:

$$\begin{aligned} \boldsymbol{u}(t,z) &\ll 16RS\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\frac{nA}{R}\right)^k \varphi^{(k)}\left(\frac{|z|}{R}\right)\mathbf{1} \\ &= 16RS\varphi\left(\frac{nAt+|z|}{R}\right)\mathbf{1}, \end{aligned}$$

as the power series above is simply the (partial) Taylor expansion of φ about $t = 0$. Since the obtained formal solution $\mathbf{u}$ is majorized by a function that is analytic in a neighborhood of the origin, we conclude that it is convergent at least in that neighborhood. This concludes the proof of convergence and of Theorem 1.

CONCLUSION

The use of Lax's majorant function $\varphi(x)$ enabled us to establish, in a very straightforward manner, the convergence of the unique formal solution (1.2) and hence the analyticity of the flow associated with an analytic vector field. Since our method does not require the introduction of new norms and auxiliary functions, it is accessible even to senior undergraduate students. The majorizing properties of $\varphi(x)$ suggest that our technique may be applicable in proving the convergence of formal solutions of other nonlinear partial differential equations.

CONFLICT OF INTEREST

All authors declare no conflicts of interest in this paper.

CONTRIBUTIONS OF INDIVIDUAL AUTHORS

Both authors equally contributed in proving the result and in writing the article.

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